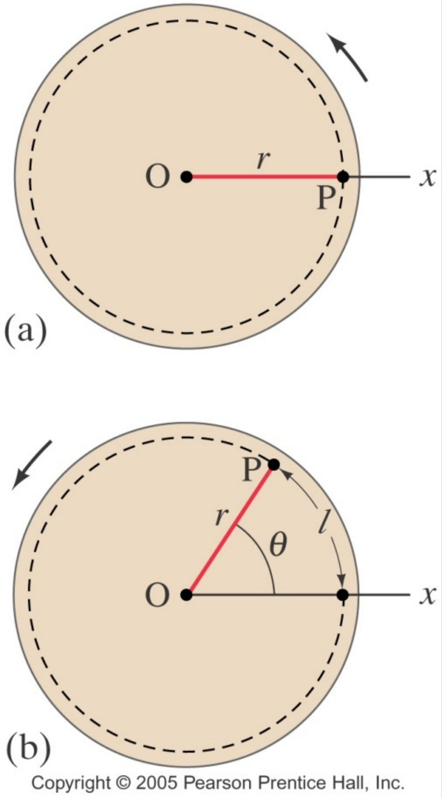
Chapter 8 – Rotational Motion

Chapter 7 – translational motion – linear momentum
Rotational motion – angular momentum
Rigid object – shape does not change



P moves in circles
Axis of rotation – where the center of circle lies
r-radius
θ−how far P has rotated
l – distance traveled, length subtended by θ

RADIAN (rad):
1 rad = angle subtended an by arc l whose length =r

In radians, angle is given by:

$1 \ \mathbf{rad} \Leftrightarrow l = r$

$$\theta = \frac{l}{r}$$

(dimensionless, but we write rad)

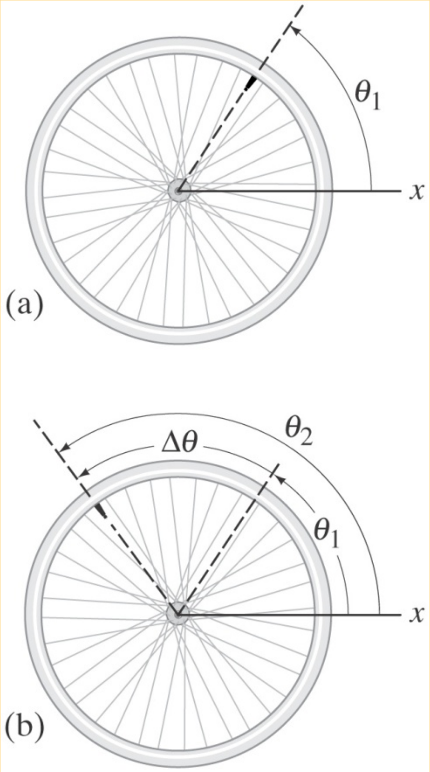
RADIANS to degrees:

$$1 \ \mathbf{rev} = 360^o = 2\pi \ \mathbf{rad}$$

Ex. 8-1 A bike wheel rotates 4.50 revolutions. How many radians has it rotated?

$$4.5 \ \mathbf{rev} = 9\pi \ \mathbf{rad} = \mathbf{28.3} \ \mathbf{rad}$$

Angular Quantities



Angular displacement: $\Delta\theta = \theta_2 - \theta_1$

Average angular velocity $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$

Instantaneous angular velocity: $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$

Unit: **rad/s**

Average angular acceleration $\bar{\alpha} = \frac{\omega_2 - \omega_1}{\Delta t} = \frac{\Delta\omega}{\Delta t}$

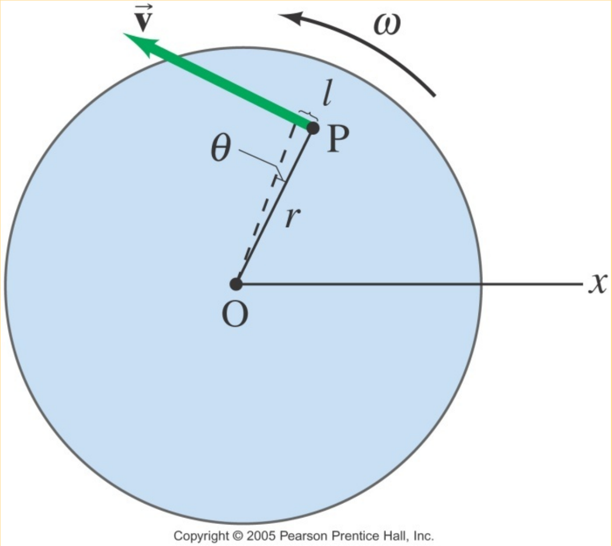
Instantaneous angular acceleration: $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$

Unit: **rad/s^2**

<u>CONVENTION:</u>
+ counterclockwise
- clockwise

ALL points in a <u>rigid object</u> rotate with the <u>same</u> angular velocity, therefore α is also the same for all points

Linear and Angular Quantities



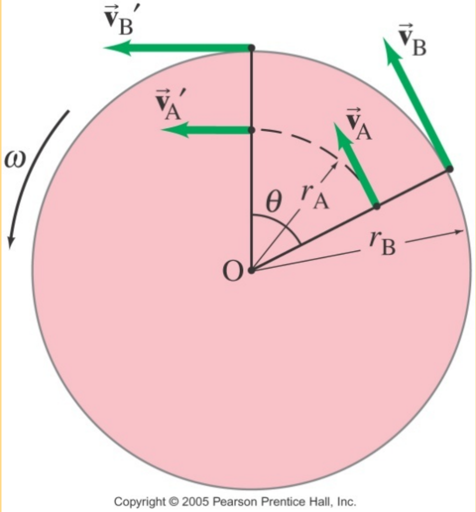
$$\theta = \frac{l}{r} \Rightarrow \Delta\theta = \frac{\Delta l}{r}$$

$$v = \frac{\Delta l}{\Delta t} = r\frac{\Delta\theta}{\Delta t} = r\omega$$

$$v = r\omega$$

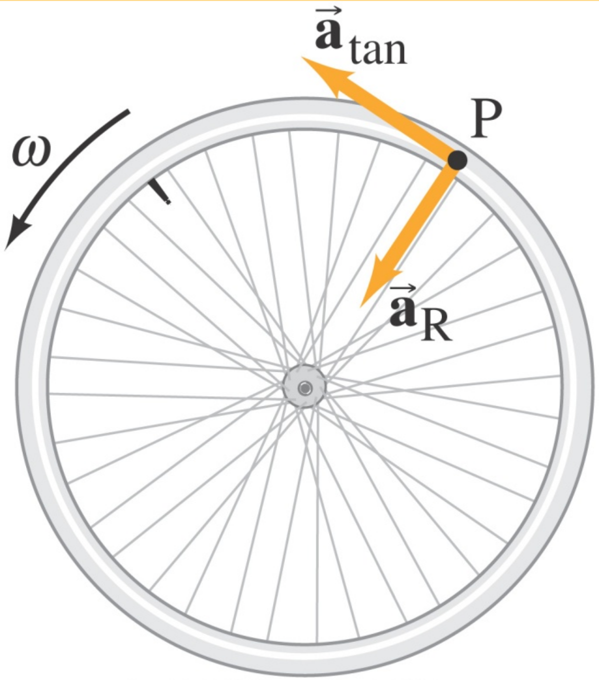
w is the same for ALL points,
it depends on the angle

v is greater for larger r,
the distance traveled increases with r



On a rotating carousel, if you sit near the outer edge your linear velocity is larger than if you sit close to the center, but w is the same for both.

Linear and Angular Quantities



If the velocity (angular/linear) of a rotating object changes, it has a **tangential acceleration**:

$$a_{tan} = \frac{\Delta v}{\Delta t} = r\frac{\Delta \omega}{\Delta t} = r\alpha$$

$$a_{tan} = r\alpha$$

Even if the angular velocity is constant, each point on the object has a **centripetal acceleration:**

$$a_R = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \omega^2 r$$

α is the same for ALL points,

a (tan or cp) is greater for larger r,

$$a = \sqrt{a_R^{\ 2} + a_{tan}^{\ 2}}$$

Linear and Angular Quantities

TABLE 8–1 Linear and Rotational Quantities

Linear	Type	Rotational	Relation
x	displacement	θ	$x = r\theta$
v	velocity	ω	$v = r\omega$
$a_{\tan}$	acceleration	α	$a_{\tan} = r\alpha$

$$\theta = \frac{l}{r}$$

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t}$$

$$\overline{\alpha} = \frac{\Delta\omega}{\Delta t}$$

$$a_R = \frac{v^2}{r} = \omega^2 r$$

Frequency and angular velocity:

$$v = \frac{2\pi r}{T} = 2\pi r f \Rightarrow f = \frac{v}{2\pi r} = \frac{r\omega}{2\pi r}$$

$$f = \frac{\omega}{2\pi}$$

$$T = \frac{1}{f}$$

Unit for frequency:

rev/s = Hz = s^{-1}

Examples

Ex.8-4 A carousel is at rest. At t=0 it is given a constant angular acceleration α=0.060 rad/s^2, which increases its angular velocity for 8.0 s. After this the angular velocity is kept constant. At t=8.0 s determine:

(a) The angular velocity of the carousel
(b) The linear velocity of a child located 2.5 m from the center
(c) The tangential acceleration
(d) The centripetal acceleration
(e) The total acceleration of the child
(f) The frequency and the period

(a) 0.48 rad/s (b) 1.2 m/s (c) 0.15 m/s^2 (d) 0.58 m/s^2
(e) 0.60 m/s^2 (f) f= 0.076 Hz and T=13 s

Constant Angular Acceleration

The equations of motion for constant angular acceleration are the same as those for linear motion, with the substitution of the angular quantities for the linear ones.

Constant angular acceleration: $\overline{\alpha} = \alpha$

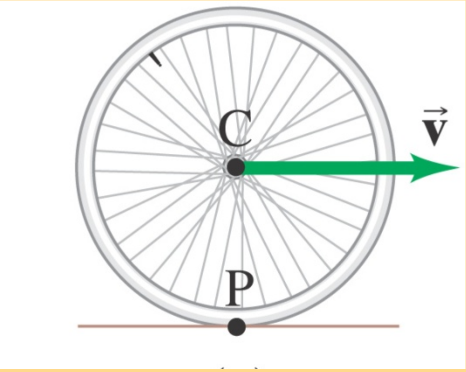
Angular	Linear
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$
$\overline{\omega} = \dfrac{\omega + \omega_0}{2}$	$\overline{v} = \dfrac{v + v_0}{2}$

Ex. 8-6 A centrifuge rotor is accelerated from rest to 20,000 rpm in 30 s. (a) What is its average angular acceleration? (b) Through how many revolutions has it turned during its acceleration period, assuming constant α?

$$\overline{\alpha} = 70 rad/s^2$$

$$5.0 \times 10^3 rev$$

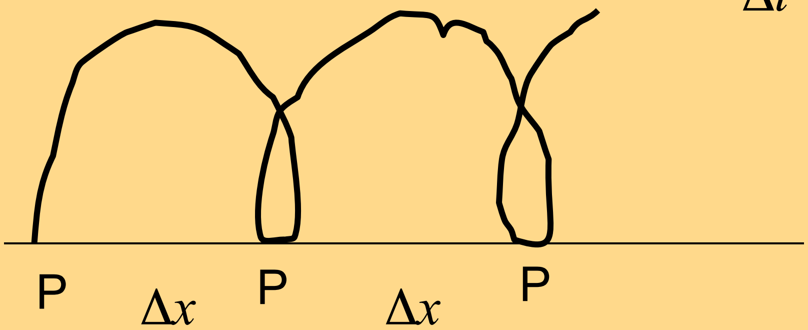
Rolling Motion (without slipping)



A wheel is rolling without slipping. The point P, touching the ground, is instantaneously **at rest**, and the center moves with velocity v.

$$v = \frac{\Delta x}{\Delta t} = \frac{2\pi r}{T} = r\omega$$

$$\boxed{v = r\omega}$$



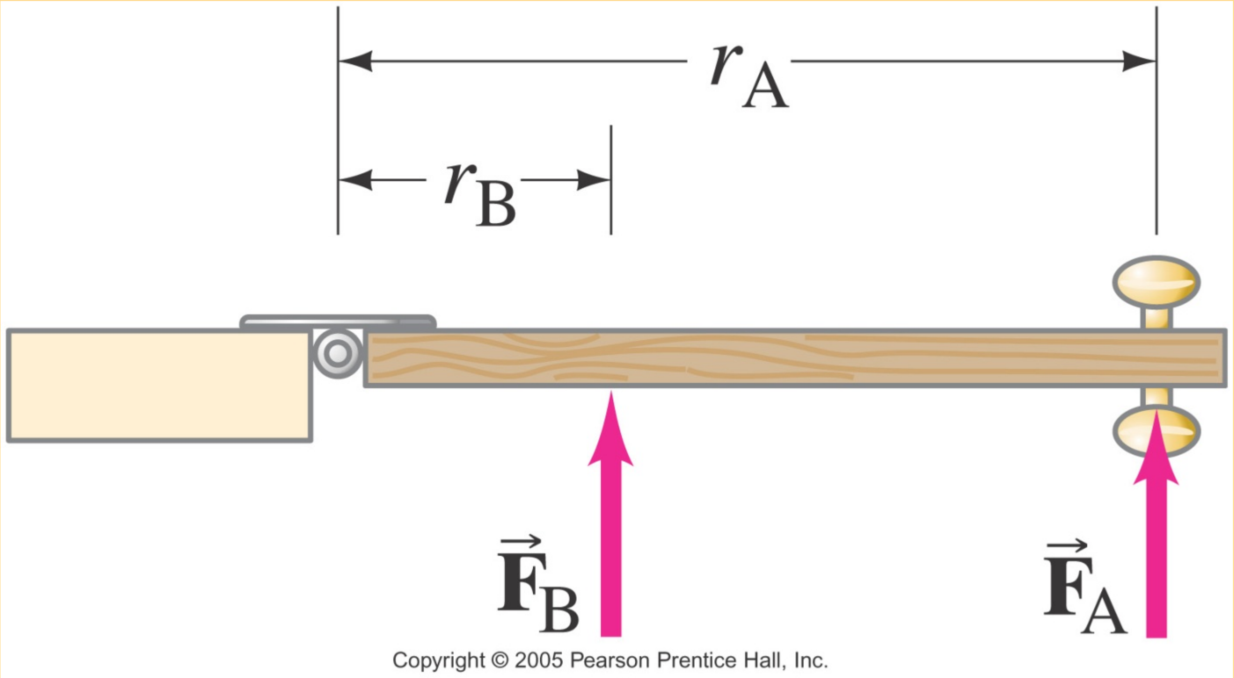
Ex. 8-7 A bicycle slows down uniformly from 8.40m/s to rest over a distance of 115 m. The tire has a diameter of 68.0 cm. Determine (a) the angular velocity of the wheels at t=0; (b) the total number of revolutions each wheel rotates before coming to rest; (c) the angular acceleration of the wheel; (d) the time it took to come to a stop.

(a) 24.7 rad/s (b) 53.8 rev (c) –0.902 rad/s^2 (d) 27.4 s

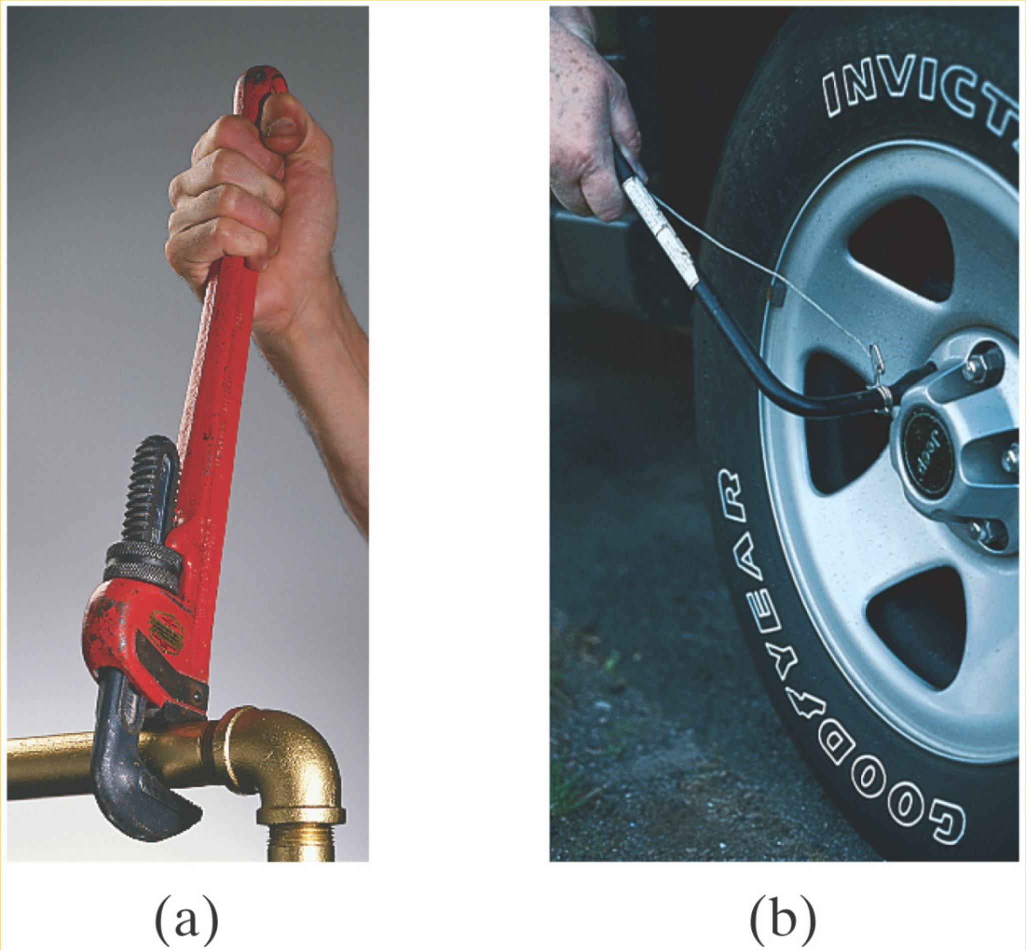
Torque

To make an object start rotating, a force is needed;
what matters: magnitude, direction and **WHERE** the force is applied

The perpendicular distance from the axis of rotation to the line along which the force acts is called the **lever arm.**



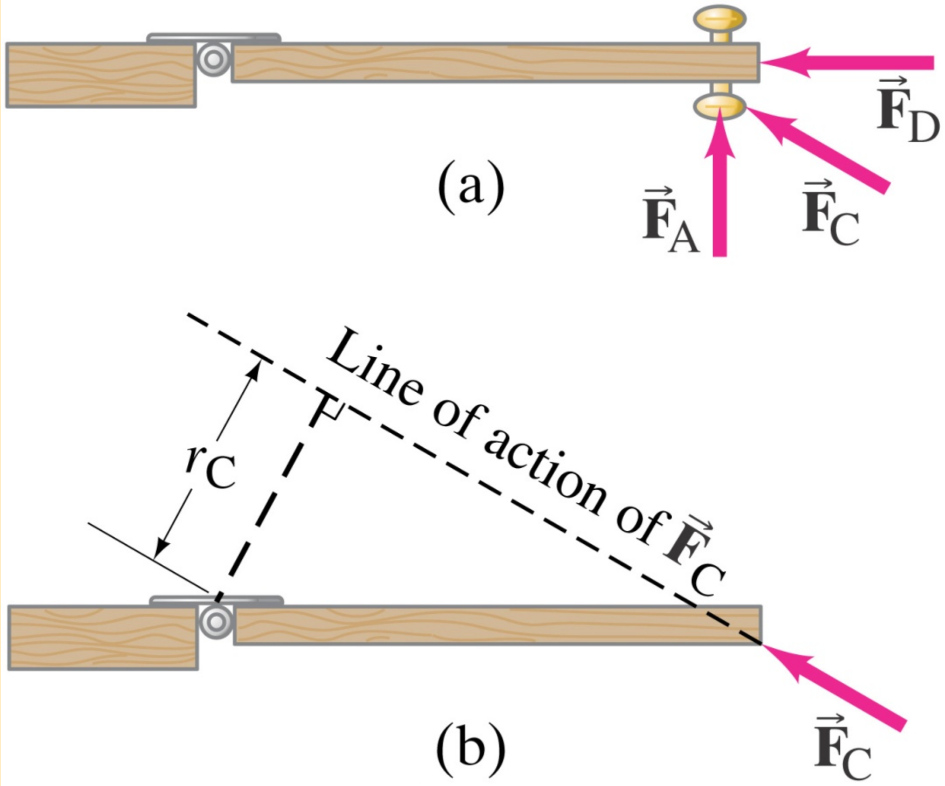
Lever arm



A longer lever arm is very helpful in rotating objects

Lever arm

The lever arm for F_A is the distance from the knob to the hinge; the lever arm for F_D is zero; and the lever arm for F_C is as shown.



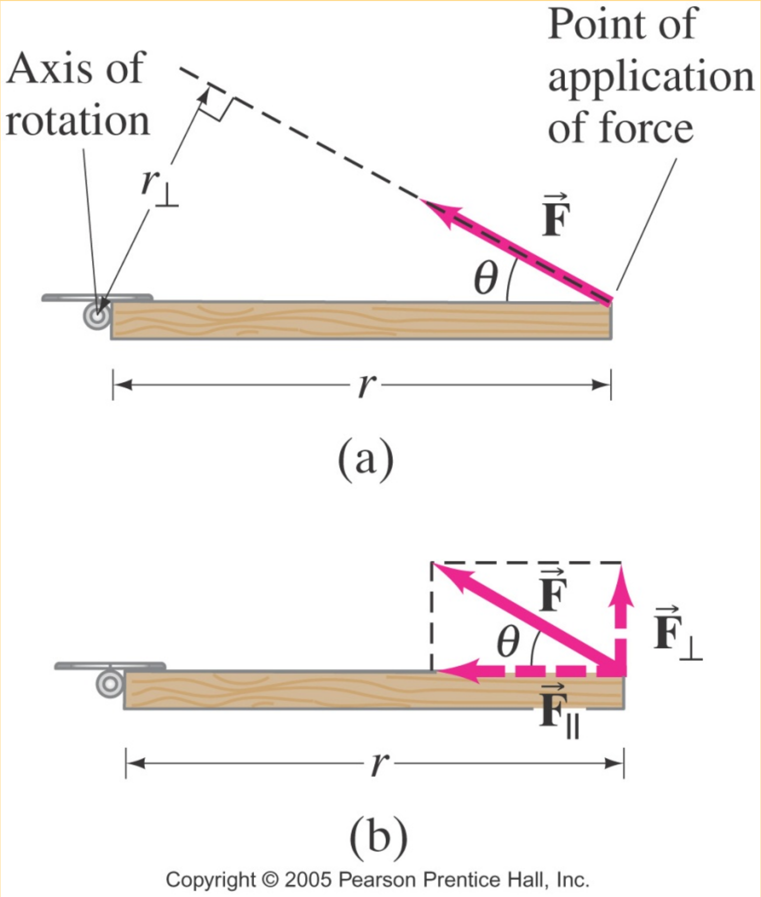
The perpendicular distance from the axis of rotation to the line along which the force acts is called the **lever arm.**

The angular acceleration is proportional to the product of the forces times the lever arm (also called moment arm)

This product = **TORQUE** or moment of the force (τ)

$\alpha \propto \tau$ in analogy with $a \propto F$

Torque



Torque is defined as:

$$\tau = r_\perp F$$

or equivalently:

$$\tau = r F_\perp$$

Unit:
SI: **m.N**
Careful: J only for energy
CGS: cm.dyne

By decomposing the forces:
The parallel one exerts no torque (r=0)

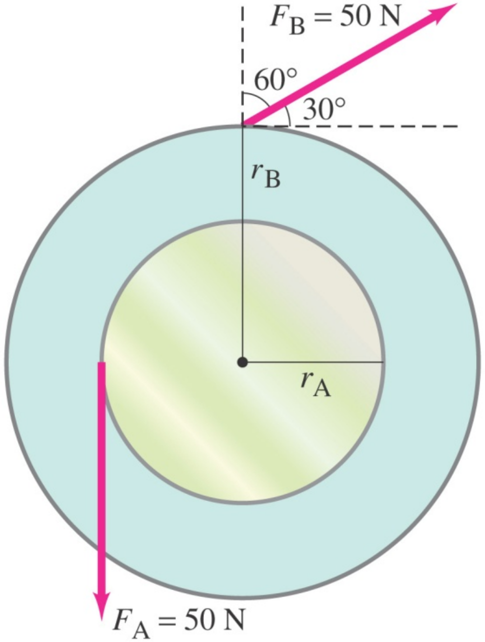
$$\left.\begin{aligned} F_\perp &= F\sin\theta \\ r_\perp &= r\sin\theta \end{aligned}\right\} \quad \tau = Fr\sin\theta$$

Net Torque

More than one torque acts on an object: angular acceleration is proportional to the NET torque.

CONVENTION: **+** sign for torque that acts to rotate the object **counterclockwise**
- for **clockwise**

Ex. 8-9 Two thin disk-shaped wheel of radii r_A=30 cm and r_B=50 cm are Attached to each other on an axle that passes through the center of each. Calculate the net torque on this compound wheel due to the two forces, each of magnitude 50 N.



- 6.7 m.N

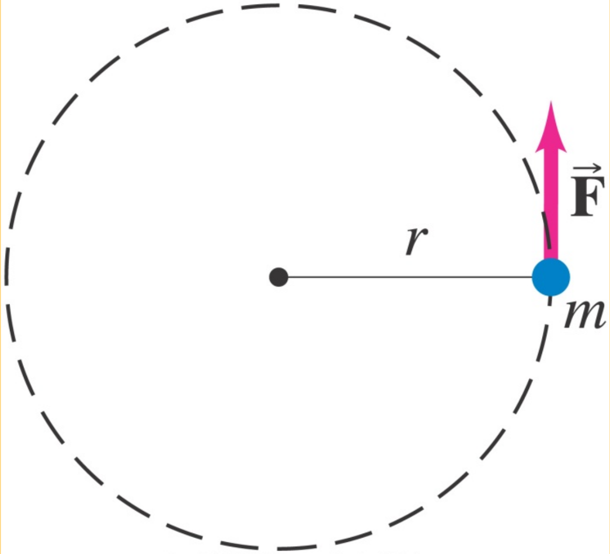
Torque and Rotational Inertia

$$\sum F = ma$$

Newton's 2nd law

$$\sum \tau = ??\alpha$$

Net torque



Tangential acceleration:

$$F = ma = mr\alpha$$

$$\tau = rF = mr^2\alpha$$

Rotational inertia or moment of inertia

Rigid object:
Sum for each particle

$$\sum \tau = \left(\sum mr^2\right)\alpha$$

Moment of inertia depends on the mass AND its distribution

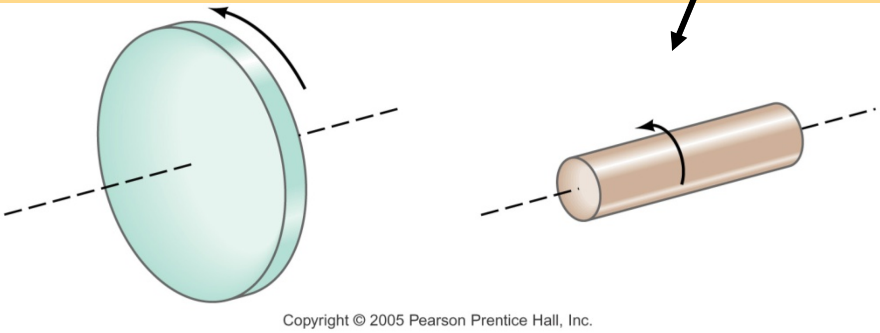
$$I = \sum mr^2 = m_1 r_1^2 + m_2 r_2^2 + \ldots$$

$$\sum \tau = I\alpha$$

Rotational Inertia/Moment of Inertia

$$I = \sum mr^2 = m_1 r_1^2 + m_2 r_2^2 + \ldots$$

These two objects have the same mass, but the one on the left has a greater rotational inertia - much of its mass is far from the axis of rotation, it is harder to start rotating it and to stop it.

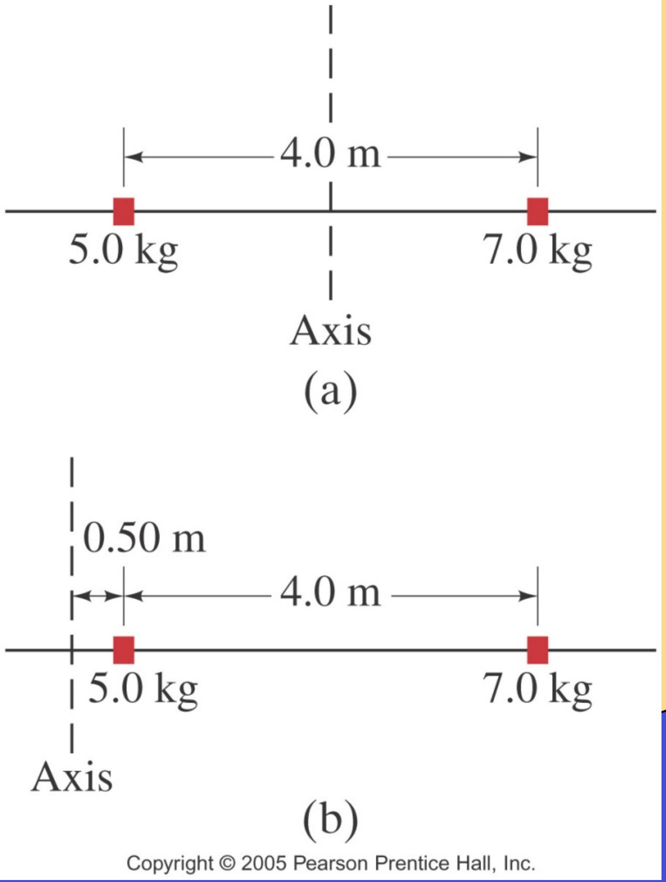


Ex. 8-10 Calculate the moment of inertia of the system (a) and (b)

$$(a) I = 48 kg.m^2$$

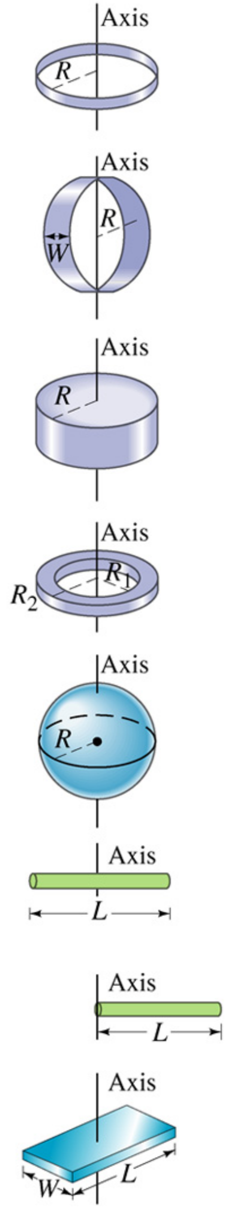
$$(b) I = 143 kg.m^2$$

I changes with the axis of rotation
Mass close to the axis – little contribution



Moment of Inertia

	Object	Location of axis	Moment of inertia
(a)	**Thin hoop,** radius R	Through center	MR^2
(b)	**Thin hoop,** radius R width W	Through central diameter	$\frac{1}{2}MR^2 + \frac{1}{12}MW^2$
(c)	**Solid cylinder,** radius R	Through center	$\frac{1}{2}MR^2$
(d)	**Hollow cylinder,** inner radius R_1 outer radius R_2	Through center	$\frac{1}{2}M(R_1^2 + R_2^2)$
(e)	**Uniform sphere,** radius R	Through center	$\frac{2}{5}MR^2$
(f)	**Long uniform rod,** length L	Through center	$\frac{1}{12}ML^2$
(g)	**Long uniform rod,** length L	Through end	$\frac{1}{3}ML^2$
(h)	**Rectangular thin plate,** length L, width W	Through center	$\frac{1}{12}M(L^2 + W^2)$

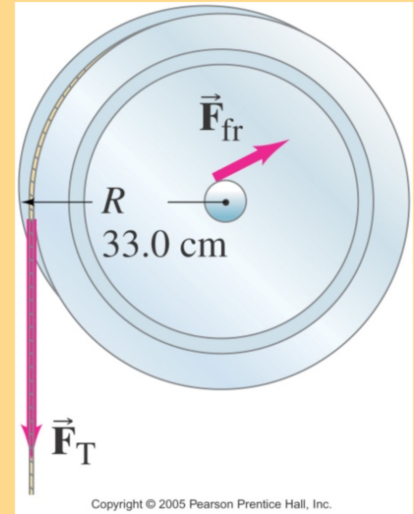


The rotational inertia of an object depends not only on its mass distribution but also the location of the axis of rotation – compare (f) and (g), for example.

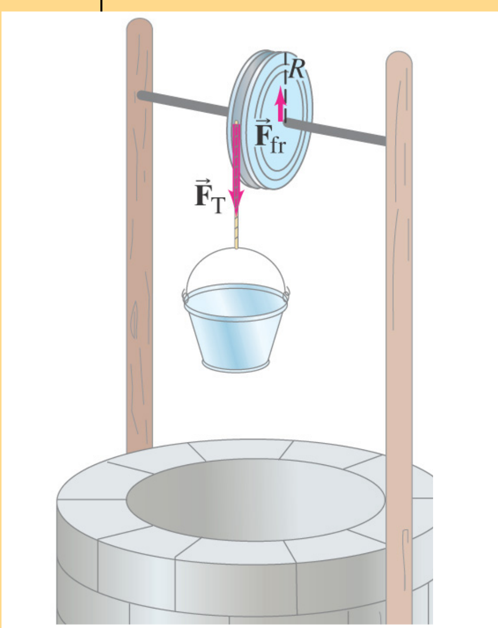
Examples

Ex. 8-11 A 15.0-N force is applied to a cord wrapped around a pulley of M=4.00 kg and radius R=33.0 cm. The pulley accelerates uniformly from rest to an angular speed of 30.0 rad/s in 3.00 s. If there is a frictional torque 1.10 m.N at the axle, determine the moment of inertia of the pulley.

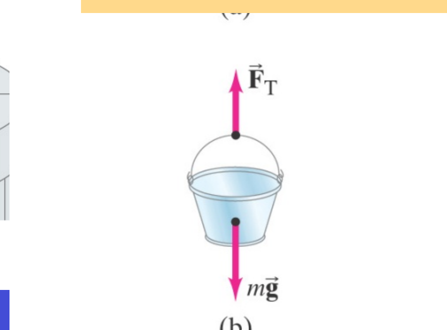
I=0.385 kg.m^2



Ex. 8-12 Same pulley (I=0.385 kg.m^2), but now we have a bucket of weight 15.0 N hanging from the cord. Calculate the angular acceleration of the pulley and the linear acceleration of the bucket.



(a)



(b)

α=6.98 rad/s^2

a = R α

Rotational Kinetic Energy

Translational Kinetic Energy: $K = \frac{1}{2}mv^2$

Rigid Object:
Total kinetic energy is the sum for each particle $K = \sum \frac{1}{2}mv^2 = \sum \frac{1}{2}mr^2\omega^2 = \frac{1}{2}I\omega^2$

$$\text{rotational KE} = \tfrac{1}{2}I\omega^2$$

(unit is J)

A object that has both translational and rotational motion also has both translational and rotational kinetic energy:

$$\text{KE} = \tfrac{1}{2}Mv_{CM}^2 + \tfrac{1}{2}I_{CM}\,\omega^2$$

I_{CM} is the moment of inertia about an axis through the CM,
w is the angular velocity about this axis, M is the total mass of the object,
v_{CM} is the linear velocity of the CM

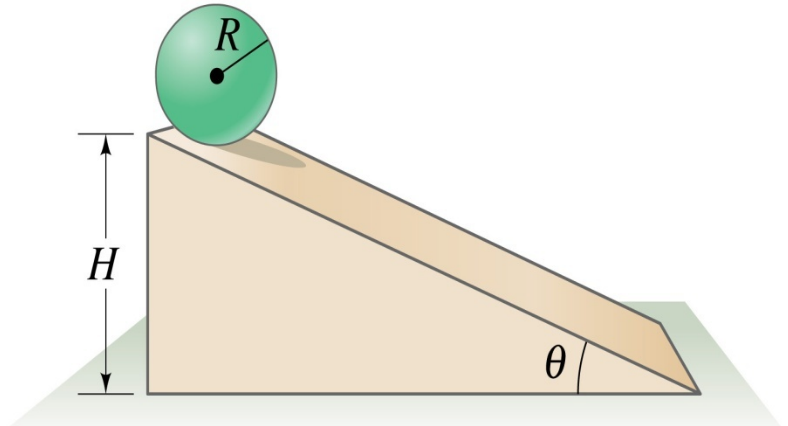
Example

$$KE = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

Ex. 8-13 What will be the speed of a solid sphere of mass M and radius R when it reaches the bottom of an incline it it starts from rest at a vertical height H and rolls without slipping? (solid sphere: I_{CM}=2/5 MR^2) Compare the result with that of an object sliding down a frictionless incline.

$$v = \sqrt{\frac{10}{7}gH} \quad (rolling)$$

$$v = \sqrt{2gH} \quad (sliding)$$



An object sliding with no friction or rotation transforms its initial potential energy entirely into translational kinetic energy

Work done by torque:

$$W = F\Delta l = Fr\Delta\theta$$

$$\tau = rF$$

$$W = \tau\Delta\theta$$

Angular Momentum

By analogy: $\frac{1}{2}mv^2 \rightarrow \frac{1}{2}I\omega^2$

$p = mv \rightarrow L = I\omega$ angular momentum, units: $kg.m^2/s$

$$\sum F = \frac{\Delta p}{\Delta t} \rightarrow \sum \tau = \frac{\Delta L}{\Delta t}$$

$$\sum F = ma \rightarrow \sum \tau = I\alpha$$

(m – constant) (I – constant)

$$\sum \tau = \frac{\Delta L}{\Delta t} = I\frac{(\omega - \omega_0)}{\Delta t} = I\alpha$$

Conservation of Angular Momentum

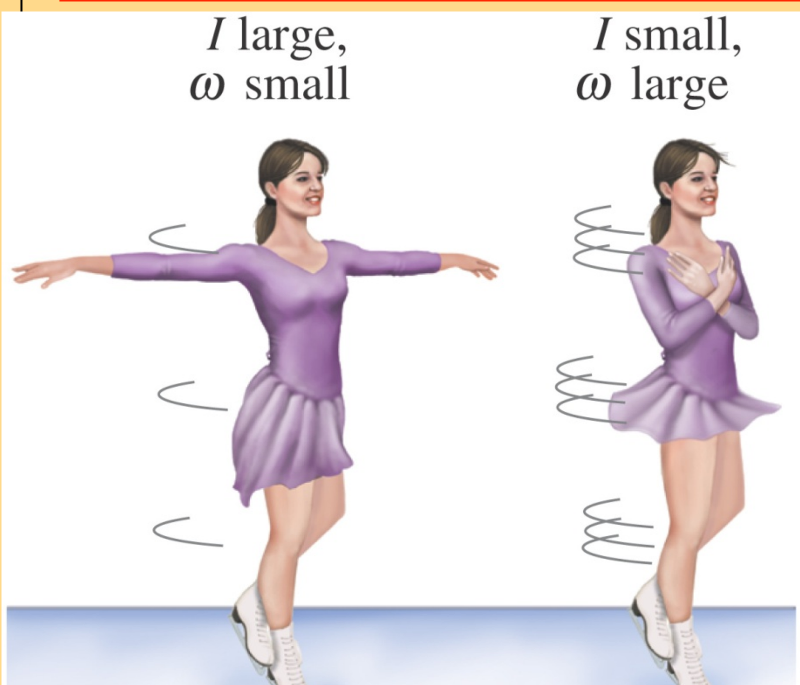
$$\sum \tau = \frac{\Delta L}{\Delta t}$$

$$I = \sum mr^2$$

If the net torque on an object is zero, the total angular **momentum is constant.**

$$I\omega = I_0\omega_0 = \text{constant}$$

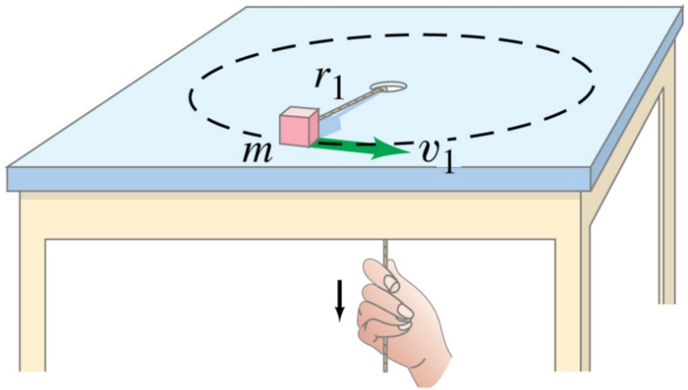
If **I** decreases then ω has to increase



Does her rotational kinetic energy increase? Where does the energy come from?

Examples

Ex. 8-15 A small mass m attached to the end of a string revolves in a circle on a frictionless tabletop (see figure). Initially it revolves at 2.4m/s in a circle of radius 0.80 m. The string is pulled through the hole so that the radius becomes 0.48 m. What is now the speed?



4.0 m/s

6x10^3 rev/s

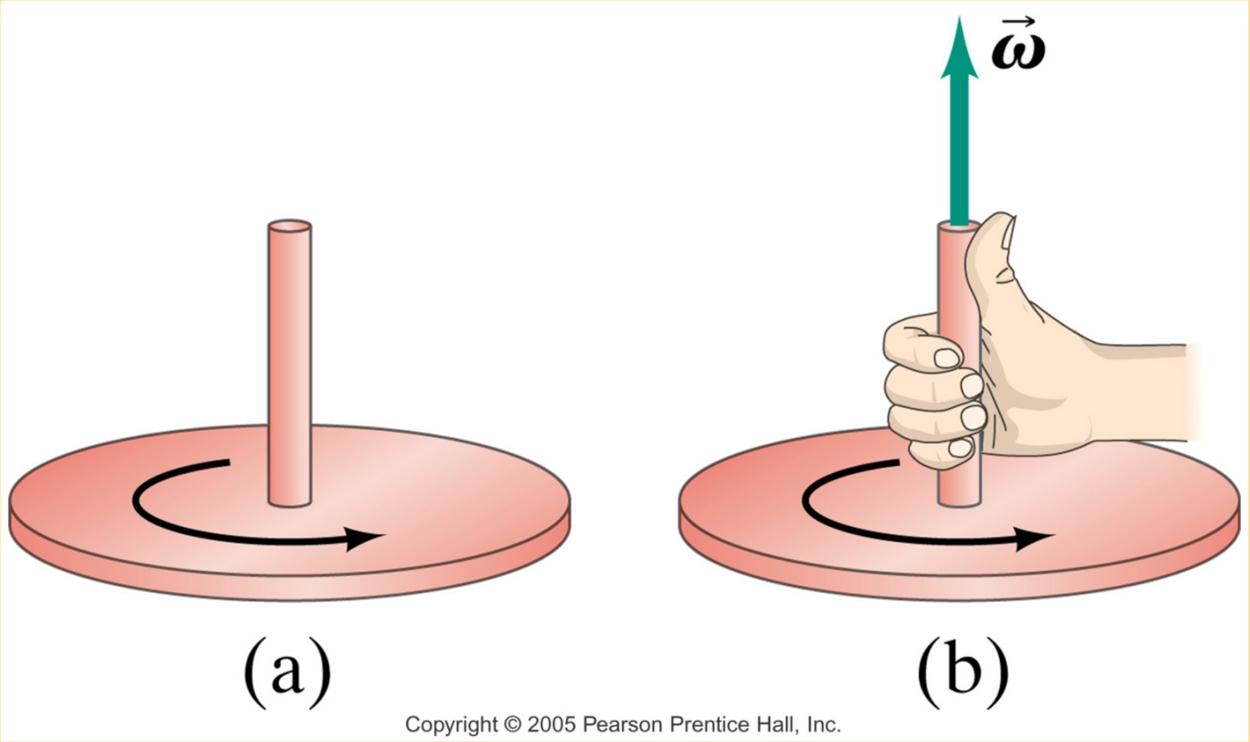
Ex. 8-16 Neutron stars are formed from the inner core of a large star that collapses due to its own gravitation to a star of a very small radius and very high density - neutron star. Neutron stars rotate extremely fast.

Suppose the core of such a star is the size of the sun (R ~ 7x10^5 km) with mass 2.0 times as great as the sun and it is rotating at 1.0 rev every 10 days. If it collapses to 10 km, what is the new frequency?

Assume the star is a uniform sphere (I_{CM}=2/5 M R^2)

Angular Momentum is a VECTOR

The angular velocity vector points along the axis of rotation; its direction is found using a right hand rule:



$$\vec{L} = I\vec{\omega}$$